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IDENTIFIABILITY PROBLEMS RELATED WITH CREDIBILITY: A SURVEY WITH APPLICATIONS

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Abstract

Premium computation in a Bayesian context requires the use of a prior distribution that the unknown risk parameter of the likelihood follows in the heterogeneous population. Sometimes, the Bayes premium is expressed as a weighted sum of the sample mean and the collective premium, known in the literature as credibility formula.

In this paper, some connections between credibility theory and identifiability problems are reviewed and modestly extended by identifying the prior distribution under different likelihoods by the form of the Bayes premium, which results under appropriate likelihood and prior distribution a credibility formula. Results under the net premium principle for Poisson, binomial, and negative binomial likelihood functions and under the Esscher premium principle for Poisson likelihood function are shown. The methodology is applied to generate a wide spectrum of discrete distributions when non-credibility formulae appear.

1. Introduction

Credibility theory provides a tool to compute premiums calculated by combining the sample information together with collateral information by

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incorporating a prior distribution to the unknown risk parameter. Assuming that the individual risk, X, has a density $f(x / \theta)$, indexed by the unknown risk parameter $\theta \in \Theta$, which has a prior distribution with density $\pi(\theta)$. Let us suppose $\pi(\theta / k)$ is the posterior density when a sample $\underline{X} = (X_1, \ldots, X_n)$ of size n is observed, $k = \sum_{i=1}^{n} X_i \in \mathbb{N}$.

A premium calculation principle (e.g., Bühlmann and Gisler [4], Calderín et al. [6], Furman and Zitikis [7], and Heilmann [14]) assigns to each risk parameter θ, a premium within the set $\mathcal{H}(f) \in \mathbb{R}$, the action space. Let $\mathcal{L} : \Theta \times \mathcal{H}(f) \to \mathbb{R}$ be a loss function that assigns to any $(\theta, H(f)) \in \Theta \times \mathcal{H}(f)$, the loss sustained by a decision-maker, who takes the action $H(f)$ and is faced with the outcome θ of a random experience. The premium must be determined such that the expected loss is minimized.

$\mathcal{L}$ is usually taken as the weighted squared-error loss function, i.e., $\mathcal{L}(a, x) = w(x)(x - a)^2$, for some function $w(x)$. See Furman and Zitikis [7], Heilmann [14], and Kamps [19] for details.

In this case, the unknown premium $H(f) \equiv P(\theta)$, called the risk premium, is given by $P(\theta) = E[Xw(X)] / E[w(X)]$, where the expectation is taken under the probability density function $f(x / \theta)$. This last expression includes some of the well known premium calculation principles: the net premium $(w(x) = 1)$, the variance premium $(w(x) = x)$, and the Esscher premium $(w(x) = \exp\{\alpha x\}, \alpha > 0)$, among others.

If experience is not available, the actuary computes the collective premium, P_C, which is given by minimizing the risk function, i.e., minimizing $E[\mathcal{L}(P(\theta), \theta)]$, where the expectation is now taken under the prior $\pi(\theta)$. Using the weighted squared-error loss function considered above, the collective premium is given by $P_C = E[P(\theta)w(P(\theta))]/E[w(P(\theta))]$.

On the other hand, if experience is available, the actuary takes a sample $\underline{X}$ the unknown risk premium can be estimated through the

Bayes (experience rated) premium P_B. This is obtained in the same way that the collective premium, but now the expectation is taken with respect to the posterior distribution $\pi(\theta / k)$.

Sometimes, it is possible to write the Bayes premium as a weighted sum of the sample mean and the collective premium, the premium to be charged to a group of policyholders in a portfolio. The weighted factor is referred to as the credibility factor and therefore, the premium obtained adopts this suggestive expression:

$$P_B = z_n l(\overline{X}) + (1 - z_n) P_C, \tag{1}$$

for some function of the sample mean $l(\overline{X})$, where $\overline{X}$ is the sample mean, P_C the collective premium, and z_n the credibility factor satisfying $z_n \in (0, 1)$, $\lim_{n \to 0} z_n = 0$, and $\lim_{n \to \infty} z_n = 1$. Some historical references on credibility theory are Gerber and Arbor [8], Herzog [15], Heilmann [14], and Gómez-Déniz [10], among others.

On the other hand, in the statistical literature, the identification problem is connected with one to one correspondence between the regression function $m(k) = \mathbb{E}(k|\theta)$ and the distributions of Θ and X. Some important contributions in this field are Cacoullos and Papageorgiou [5], Gupta and Wesolowski [12, 13], Johnson [17], Papageorgiou [22], Papageorgiou and Wesolowski [23], and Wesolowski [25], among others. In Sapatinas [24], the identification problem is studied in the context of the power-series and Poisson-Lindley distributions for both the univariate and multivariate cases. In Gupta and Wesolowski [12, 13], the identification problem is analyzed under uniform mixtures.

Nevertheless, in the actuarial settings, the posterior magnitude, called the *Bayes premium*, usually does not coincide with the posterior mean of the parameter. Therefore, we consider that this topic about indentifiability of Bayes premium has never been analyzed in the actuarial literature.

In this paper, some connections between credibility theory and identifiability problems are reviewed and modestly extended by identifying the prior distribution under different likelihoods by the form of the Bayes premium, which results under appropriate likelihood and prior distribution a credibility formula. This will be done by an one to one correspondence between the likelihood and the prior distribution, if the credibility expression is given. Results under the net premium principle for Poisson, binomial, and negative binomial likelihood functions and under the Esscher premium principle for Poisson likelihood function are shown. The methodology is applied to generate a wide spectrum of discrete distributions when non-credibility formulae appear.

In the present paper, Section 2 includes the results for the Poisson, negative binomial, and binomial cases, respectively. For these cases, the net premium principle was the only premium principle considered and identification of bonus-malus rates are also studied. In Section 3, we extend our study to the Poisson-gamma model under Esscher premium principle. Some applications are provided in Section 4 and conclusions are given in the last section.

2. The Results

In this section, we deal with the identification of the prior distribution that generates a Bayes premium under the net premium principle. This expression is linear with respect to the observed data and it is going to be determined from two sample models, the negative binomial, and the binomial. Previously, the Poisson case will be analyzed. The fact that the Bayes net premium is linear with respect to the data leads us to an expression of the premium known in actuarial settings as credibility formula.

2.1. The Poisson case

Let X be a random variable with the Poisson probability mass function, i.e., $f(x|\theta) \propto e^{-\theta}\theta^{x}$, $\theta > 0$, $x = 0, 1, \ldots$. In this case, the net risk premium is given by $P(\theta) = \theta$. The fact that the regression of X on k is linear, i.e., a credibility formula, was proved by Johnson [17]. In this section, we reproduce the proof in an alternative way.

Theorem 1. *Let X be a Poisson distribution with parameter $\theta > 0$ and $\theta \in (0, \infty)$ is a continuous random variable with density $\pi(\theta)$. Then* *the Bayes net premium,* $P_B = \mathbb{E}_{\pi(\theta/k)}(\theta)$, $k = \sum_{i=1}^{n} X_i$, *determines uniquely the distributions of* X *and* θ.

Proof. By applying Bayes' theorem, we have

$$m(k) = \mathbb{E}_{\pi(\theta/k)}(\theta) = \frac{\prod_{i=1}^{n}(x_i + 1/n)!}{\prod_{i=1}^{n} x_i!} \frac{1}{f(k)} \int_0^\infty e^{-n\theta}\theta^{k+1}\pi(\theta)\, d\theta$$

$$= \frac{\prod_{i=1}^{n}(x_i + 1/n)!}{\prod_{i=1}^{n} x_i!} \frac{f(k+1)}{f(k)}.$$

Therefore,

$$f(k+1) = \frac{\prod_{i=1}^{n} x_i!}{\prod_{i=1}^{n}(x_i + 1/n)!} m(k)f(k). \tag{2}$$

Since, this last expression corresponds to a first-order difference equation in $f(k)$, a unique solution exists. Thus, the distribution of X is uniquely determined by the function m, and therefore $\pi(\theta)$ is unique. □

It is known (see Heilmann [14]) that assuming a Poisson distribution for the risk X and Θ has the Pearson Type III distribution (gamma distribution) with the density $\pi(\theta) \propto \theta^{a-1}e^{-b\theta}$, $a > 0$, $b > 0$, the Bayes net premium is a credibility formula as in (1), where $l(x) = x$, $z_n = n/(b+n)$ and $P_C = \mathbb{E}(\Theta) = a/b$.

Therefore, as a consequence of Theorem 1, we have the next result.

Corollary 1. *If the random variable X follows a Poisson distribution with parameter* $\theta > 0$, *the only form of prior probability density function satisfying that the Bayes net premium takes the form* (1), *is the Pearson Type* III *distribution* $\pi(\theta) \propto \theta^{a-1}e^{-b\theta}$.

Finally, it is well-known that the mean of the predictive distribution coincides with the posterior mean when $\mathbb{E}(y|\theta) = \theta$, see Herzog [15] for details. Therefore, we have the following result.

Corollary 2. *If the random variable X follows a Poisson distribution with parameter* $\theta > 0$, *then the prior distribution of* θ *is uniquely determined by the mean of the predictive distribution* $f(y|k) = \int_0^\infty f(y|\theta)\pi(\theta|k)\, d\theta$.

Proof. It follows directly by applying Theorem 1 having into account that for the Poisson distribution $\mathbb{E}(y|\theta) = \theta$. □

2.2. The binomial case

Let us suppose that X is now a random variable with the binomial probability mass function, $f(x|\theta) = \binom{N}{x}\theta^x(1-\theta)^{N-x}$, $\theta > 0$, $x = 0, 1, \ldots, N$. In this case, the net risk premium is $P(\theta) = N\theta$. Then we have the next result.

Theorem 2. *Let X be a binomial distribution with parameters* $N > 0$, $0 < \theta < 1$, *and* θ *is a continuous random variable with density* $\pi(\theta)$. *Then the Bayes net premium,* $P_B = \mathbb{E}_{\pi(\theta/k)}(N\theta)$, $k = \sum_{i=1}^n X_i$, *determines uniquely the distributions of X and* θ.

Proof. By applying Bayes' theorem, we have

$$m(k) = \mathbb{E}(P(\theta)|k) = \frac{N}{f(k)}\int_0^1 \prod_{i=1}^n \binom{N}{x_i}\theta^{k+1}(1-\theta)^{nN-(k+1)}(1-\theta)\pi(\theta)\, d\theta$$

$$= \frac{N}{f(k)}\left[\int_0^1 \prod_{i=1}^n \binom{N}{x_i}\theta^{k+1}(1-\theta)^{nN-(k+1)}\pi(\theta)\, d\theta\right.$$

$$\left. - \int_0^1 \theta \prod_{i=1}^n \binom{N}{x_i}\theta^{k+1}(1-\theta)^{nN-(k+1)}\pi(\theta)\, d\theta\right]$$

$$= \frac{N\prod_{i=1}^{n}\binom{N}{x_i}}{\prod_{i=1}^{n}\binom{N}{x_i + 1/n}} \frac{f(k+1)}{f(k)}[1 - m(k+1)].$$

Therefore,

$$f(k+1) = \frac{\prod_{i=1}^{n}\binom{N}{x_i + 1/n}}{N\prod_{i=1}^{n}\binom{N}{x_i}} \frac{m(k)}{1 - m(k+1)} f(k), \tag{3}$$

and due to Theorem 1, the assertion holds. □

Note that taking in (3), $n = 1$ and $k = x$, we have obtained as a particular case result (2.5) in Theorem 2.1 in Papageorgiou [22].

$$\frac{f(x+1)}{f(x)} = \frac{N - x}{N(x+1)} \frac{m(x)}{1 - m(x+1)}. \tag{4}$$

If we assume a binomial distribution for the risk X and θ has the beta distribution $\pi(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$, $a > 0$, $b > 0$, the Bayes net premium is a credibility formula as in (1), where $l(x) = x$, $z_n = Nn/(Nn + a + b)$, and $P_C = \mathbb{E}(N\Theta) = Na/(a + b)$.

Therefore, as a consequence of Theorem 2, we have the following result.

Corollary 3. *If the random variable X follows a binomial distribution, the only form of prior probability density function satisfying that the Bayes net premium takes the form* (1), *is the beta distribution* $\pi(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$, $a > 0$, $b > 0$.

2.3. The negative binomial case

Let us suppose that X is now a random variable with the negative binomial probability mass function $f(x|\theta) = \binom{r + x - 1}{x}\left(\frac{r}{r+\theta}\right)^r\left(\frac{\theta}{r+\theta}\right)^x$, $\theta > 0$, $r > 0$, $x = 0, 1, \ldots$. This parameterization of the negative binomial model in the actuarial context has been considered by Gómez-Déniz and

Vázquez [9] and Meng et al. [21], among others. In this case, the net risk premium is given by $P(\theta) = \theta$ and we have next result.

Theorem 3. *Let X be a negative binomial distribution with parameters $r, \theta > 0$, and $\theta \in (0, \infty)$ a continuous random variable with density $\pi(\theta)$. Then the Bayes net premium, $P_B = \mathbb{E}_{\pi(\theta/k)}(\theta)$, $k = \sum_{i=1}^n X_i$, determines uniquely the distributions of X and θ.*

Proof. By applying Bayes' theorem, we have

$$\begin{aligned}
m(k) &= \mathbb{E}(P(\theta)|k) \\
&= \frac{1}{f(k)} \int_0^\infty \prod_{i=1}^n \binom{r + x_i - 1}{x_i} r^{nr} \theta^{k+1} (r+\theta)^{-(nr+k+1)} (r+\theta)\pi(\theta)\, d\theta \\
&= \frac{1}{f(k)} \Bigg[r \int_0^\infty \prod_{i=1}^n \binom{r + x_i - 1}{x_i} r^{nr} \theta^{k+1} (r+\theta)^{-(nr+k+1)} \pi(\theta)\, d\theta \\
&\quad + \int_0^\infty \prod_{i=1}^n \theta \binom{r + x_i - 1}{x_i} r^{nr} \theta^{k+1} (r+\theta)^{-(nr+k+1)} \pi(\theta)\, d\theta \Bigg] \\
&= \frac{\prod_{i=1}^n \binom{r + x_i - 1}{x_i}}{\prod_{i=1}^n \binom{r + x_i + 1/n - 1}{x_i + 1/n}} \frac{f(k+1)}{f(k)} [r + m(k+1)].
\end{aligned}$$

Therefore,

$$f(k+1) = \frac{\prod_{i=1}^n \binom{r + x_i + 1/n - 1}{x_i + 1/n}}{\prod_{i=1}^n \binom{r + x_i - 1}{x_i}} \frac{m(k)}{r + m(k+1)} f(k). \tag{5}$$

□

Observe that the choice $n = 1$ and $k = x$ in Theorem 3 give us result (2.7) in Theorem 2.1 in Papageorgiou [22] as a particular case

$$\frac{f(x+1)}{f(x)} = \frac{r+x}{x+1}\frac{m(x)}{r+m(x+1)}.$$

It is known (see Gómez-Déniz and Vázquez [9] and Meng et al. [21]) that assuming a negative binomial distribution for the risk X and θ has the generalized Pareto distribution, $\mathcal{GP}(\zeta, r, s), \zeta > 0, r > 0, s > 0,$ with the following density function

$$\pi(\theta;\, \zeta,\, r,\, s) = \frac{\Gamma(s\zeta + sr + 1)}{\Gamma(s\zeta)\Gamma(sr+1)} \frac{r^{sr+1}\theta^{s\zeta-1}}{(r+\theta)^{s\zeta+sr+1}} I_{(0,\,\infty)}(\theta), \tag{6}$$

the Bayes net premium is a credibility formula as in (1), where $l(x) = x$, $z_n = n/(s+n)$, and $P_C = \mathbb{E}(\Theta) = \zeta$.

Thus, we have the following result as a consequence of Theorem 3.

Corollary 4. *If the random variable X follows a negative binomial distribution, the only form of prior probability density function satisfying that the Bayes net premium takes the form* (1), *is the generalized Pareto distribution in* (6).

Another approach to the Bayes setup analyzed above can be found when practitioners suppose that a correct prior $\pi(\theta)$ exists, but they are unable to apply the pure Bayesian assumption, perhaps because they are not confident enough to specify it completely. Thus, a prior $\pi(\theta)$ is assigned to the risk parameter θ, which is a good approach for the true prior. A common approach to prior uncertainty in Bayesian analysis is to choose a wide class of prior distributions, and then calculate the range of Bayes actions as the prior ranges over that class. This is known as the robust Bayesian methodology (see Calderín et al. [6] and Gómez-Déniz [11]). An alternative method to this approach consists of choosing a procedure, which lies between the Bayes action and the robust Bayesian methodology. Such a hybrid approach is known as the posterior regret Γ-minimax principle. In Gómez-Déniz [11], some new credibility formula were obtained by using this methodology and, in a similar way as the one developed in this paper, it can be proved that they are unique. Furthermore, in Landsman and Makov [20], the exponential dispersion family of distribution was used to obtain credibility expressions as in (1).

This analysis generalizes the results in Jewell [16], which deals with the same study, but with the natural exponential family of distributions instead. If we observe Theorem 2 in Landsman and Makov [20], we can derive that if $\pi(\theta)$ is a maximum entropy prior for θ under the restriction $\int_0^\infty \pi(\theta)\, d\theta = \theta_0$, then the prior Bayes net premium determines uniquely the distribution of X and θ. This last result is a consequence of the fact that the Bayes net premium is a credibility expression as in (1). Since, Poisson, binomial, and negative binomial belong to the exponential dispersion family of distributions, the maximum entropy priors satisfying the restriction above are, obviously, the Pearson Type III distribution, the beta distribution and the generalized Pareto distribution, respectively. Finally, credibility expressions under the net premium principle are obtained from these distributions.

2.4. Extensions to identify bonus-malus rates

It is well-known that in bonus-malus systems, the basic premium is modified, depending on the number of claims made during the year, according to the transition rules of the system set up. We can compute bonus-malus net premiums (see Gómez-Déniz and Vázquez [9] and Meng et al. [21]) from $P_{BM} = \mathbb{E}_{\pi(\theta|k)}(P(\theta)) / \mathbb{E}_{\pi(\theta)}(P(\theta))$, which results under appropriate likelihood and prior distributions a credibility expression.

For example, if we assume a Poisson distribution for the risk X and Θ has the Pearson Type III distribution, we have that

$$P_{BM} = z_n l(\overline{X}) + (1 - z_n) l(P_C), \tag{7}$$

where $z_n = n / (b + n)$ and $l(x) = b / ax$, being $P_C = a / b$. Alternative expressions appear also for the negative binomial-beta and binomial-beta pairs of likelihood and prior distributions.

The identifiability of bonus-malus premiums can be proved in a similar way as previous theorems. Let us suppose now, unlike the previous subsections, that the bonus-malus premium is known. The knowledge of P_{BM} is not sufficient to identify it. We need to include the additional assumption that $P_C = \mathbb{E}_{\pi(\theta)}(P(\theta))$ is known.

Then Corollaries 1, 3, and 4 are valid, when Bayes premium and expression (1) are replaced by bonus-malus net premium and expression (7), respectively, and assuming that P_C is known.

3. Beyond the Squared-Error Loss Function

The loss function usually taken in Bayesian settings is the squared-error loss which produces the posterior mean of the unknown parameter as its estimate, the net premium principle in actuarial science. This case has been studied above. Nevertheless, it is worthwhile to consider other loss functions and therefore other posterior quantities. These new posterior quantities produces others premium principles, as the variance and Esscher premiums (see Heilmann [14] and Calderín et al. [6], among others).

The weighted squared error-loss function $\mathcal{L}(x, a) = e^{\alpha x}(x - a)^2$, $\alpha > 0$ generates the Esscher premium principle (see Heilmann [14] and Gerber and Arbor [8]). In this case, the risk and collective Esscher premiums are given by $P(\theta) = \mathbb{E}_{f(x|\theta)}(Xe^{\alpha X}) / \mathbb{E}_{f(x|\theta)}(e^{\alpha X})$ and $P_C = \mathbb{E}_{\pi(\theta)}[P(\theta)e^{\alpha P(\theta)}] / \mathbb{E}_{\pi(\theta)}[e^{\alpha P(\theta)}]$, respectively. The Bayes Esscher premium is similar to the collective premium by interchanging the prior distribution by the posterior distribution.

It is known (see Heilmann [14] and Gómez-Déniz [10]) that under the Esscher premium principle and likelihood-prior Poisson-Pearson Type III distribution, the Bayes Esscher premium is an exact credibility formula.

Next result provides the identifiability of the prior distribution given the Bayes Esscher credibility premium.

Theorem 4. *Let X be a Poisson distribution with parameter* $\theta > 0$ *and* $\theta \in (0, \infty)$ *a continuous random variable with density* $\pi(\theta)$. *Then the Bayes net premium,* $P_B = \mathbb{E}_{\pi(\theta|k)}[P(\theta)\, e^{\alpha P(\theta)}] / \mathbb{E}_{\pi(\theta|k)}[e^{\alpha P(\theta)}]$, $k = \sum_{i=1}^{n} X_i$, *determines uniquely the distributions of X and* θ.

Proof. The ratio of posterior means

$$P_B = \frac{\int_0^\infty P(\theta) e^{\alpha P(\theta)} \pi(\theta|k)\, d\theta}{\int_0^\infty e^{\alpha P(\theta)} \pi(\theta|k)\, d\theta} = \frac{\int_0^\infty P(\theta) e^{\alpha P(\theta)} f(k|\theta)\, \pi(\theta)\, d\theta}{\int_0^\infty e^{\alpha P(\theta)} f(k|\theta)\, \pi(\theta)\, d\theta},$$

can be rewritten as a simple posterior mean in the following way,

$$P_B = \int_0^\infty P(\theta) \pi^*(\theta|k)\, d\theta,$$

where $\pi^*(\theta)$ is the probability density function given by

$$\pi^*(\theta) = \frac{e^{\alpha P(\theta)} \pi(\theta)}{\int_0^\infty e^{\alpha P(\theta)} \pi(\theta) d\theta},$$

from which we identify the prior distribution $\pi^*(\theta)$, in the same way as in Theorem 1, and, consequently, the prior distribution $\pi(\theta)$ and the distribution of X. □

It is known (see Heilmann [14], Gómez et al. [10]) that assuming a Poisson distribution for the risk X and θ has the Pearson Type III distribution, the Bayes Esscher premium is a credibility formula as in (1), where $z_n = n / (b + n - \alpha e^{\alpha})$, $b + n > \alpha e^{\alpha}$, $l(\overline{X}) = e^{\alpha} \overline{X}$, and $P_C = \int_0^\infty \theta e^{\alpha} \pi(\theta)\, d\theta = a e^{\alpha} / (b - \alpha e^{\alpha})$, $b > \alpha e^{\alpha}$.

Then, as a consequence of Theorem 4, we have the following result.

Corollary 5. *If a random variable X follows a Poisson distribution, the only form of prior probability density function satisfying that the Bayes Esscher premium takes the form* (1), *is the Pearson Type* III *distribution* $\pi(\theta) \propto \theta^{a-1} e^{-b\theta}$, $z > 0$, $b > 0$.

4. Applications

In this section, we are interested in the characterization of discrete distributions by the form of the Bayes premium and the likelihood. For that reason, we will assume that the Bayes premium has a given format.

4.1. The Poisson case

If we take in (2), $n = 1$ and $k = x$, we have the simple first-order difference equation

$$f(x+1) = \frac{m(x)}{x+1} f(x),$$

and solving this equation $f(x)$ can be obtained easily and is given by

$$f(x) = \frac{f(0)}{x!} \prod_{i=0}^{x-1} m(i), \tag{8}$$

with $f(0)$ determined from $\sum_x f(x) = 1$.

By taking appropriately $m(i)$ in (8), we can obtain the marginal distribution $f(x)$. We provide some examples in the following list:

1. Let $m(i) = a$, a being a positive constant. The marginal distribution is a Poisson distribution with parameter $a > 0$.

2. Let $m(i) = ai + b$, $a > 0$, $b > 0$. In this case, the marginal is, obviously, the negative binomial distribution. The prior distribution $\pi(\theta)$ is the Pearson Type III distribution.

3. Let $m(i) = a / (b + i)$, $a > 0$, $b > 0$. In this example, we obtain after some algebra, the marginal distribution

$$f(x; a, b) = \frac{1}{I_{b-1}(2\sqrt{a})\Gamma(b)} \frac{1}{x!} \frac{a^{x+(b-1)/2}}{(b)_x}, \quad x = 0, 1, \ldots. \tag{9}$$

Here $I_n(z)$ represents the modified Bessel function of the first kind.

For the particular case $b = 1$, i.e., $m(i) = a/(1+i)$ expression (9), it is reduced to the simple marginal distribution

$$f(x;\, a) = \frac{1}{x!}\frac{a^x}{x!}\frac{1}{I_0(2\sqrt{a})},\ x = 0, 1, \ldots. \tag{10}$$

The distribution (10) is a particular case of the Conway-Maxwell distribution (see, for example, Ahmad [1]).

4. Let $m(i) = a/(i+1) + b$, $a > 0$, $b > 0$. In this case, the marginal distribution is

$$f(x;\, a,\, b) = \frac{b^x(1 + a/b)_x}{(x!)^2\ {}_1F_1(1 + a/b, 1, b)}, \quad x = 0, 1, \ldots, \tag{11}$$

where ${}_1F_1(a;\, c;\, x)$ represents the confluent hypergeometric function and $(s)_j$ is the Pochhammer's symbol. It is easy to show that the pmf (11) is a reparameterization of the confluent hypergeometric distribution in Bhattacharya [3]. This confluent hypergeometric distribution contains as particular case the Poisson distribution and the hyper-Poisson distribution in Bardwell and Crow [2].

5. $m(i) = (a+i)/(b+i)$, $a > 0$, $b > 0$. The marginal distribution is:

$$\begin{aligned} f(x;\, a,\, b) &= \frac{1}{{}_1F_1(a, b, 1)}\frac{1}{x!}\frac{(a)_x}{(b)_x} \\ &= \frac{\Gamma(b)\Gamma(a+x)}{\Gamma(a)\Gamma(b+x)}\frac{1}{{}_1F_1(a, b, 1)}, \quad x = 0, 1, \ldots. \end{aligned}$$

This distribution is also a particular case of the confluent hypergeometric distribution in Bhattacharya [3].

4.2. The binomial case

By solving the first-order difference Equation (4), we get

$$f(x) = f(0)N\binom{N}{x}\prod_{i=0}^{x-1}\frac{m(i)}{1 - m(i+1)}, \quad x = 0, 1, \ldots.$$

Let us take different values for $m(i)$:

1. Let $m(i) = a$, a being a positive constant. The marginal distribution is

$$f(x;\, N,\, a) = \binom{N}{x}\left(\frac{a}{1-a}\right)^x (1-a)^N, \quad x = 0, 1, \dots, N,$$

which corresponds to a binomial distribution with parameters N and a.

2. Let $m(i) = ai + b$, $a > 0$, $b > 0$. In this case, the prior $\pi(\theta)$ is the beta distribution and the marginal is, obviously, the beta-binomial distribution.

3. Let $m(i) = a/(b+i)$, $a > 0$, $b > 0$. The marginal distribution is:

$$f(x;\, N,\, a,\, b) = \mathcal{C}\binom{N}{x}\frac{a^x (b+1)_x}{(b)_x (b-a+1)_x}, \quad b > a-1,\ x = 0, 1, \dots, N,$$

where $\mathcal{C}$, the normalization constant, is given by

$$\mathcal{C} = \frac{1}{{}_2F_2(b+1, -N;\, b, b-a+1;\, -a)}.$$

4. Let $m(i) = (a+i)/(b+i)$, $a > 0$, $b > 0$. The marginal distribution is:

$$f(x;\, N,\, a,\, b) = \mathcal{C}\binom{N}{x}\frac{(b-a)^{-x}(a)_x (b+1)_x}{(b)_x}, \quad b > a,\ x = 0, 1, \dots, N,$$

where $\mathcal{C}$, the normalization constant, is given by

$$\mathcal{C} = \frac{1}{{}_3F_1(a, 1+b, -N : b, 1/(a-b))}.$$

4.3. The negative binomial case

By solving the first-order difference Equation (5), we get

$$f(x) = f(0)\binom{r+x-1}{x}\prod_{i=0}^{x-1}\frac{m(i)}{r+m(i+1)}, \quad x = 0, 1, \dots.$$

Again, by taking different values for $m(i)$, we have:

1. Let $m(i) = a$, a being a positive constant. The marginal distribution is

$$f(x;\, r,\, K) = \binom{r + x - 1}{x}\left(\frac{a}{a + r}\right)^x \left(\frac{r}{a + r}\right)^r, \quad x = 0, 1, \ldots,$$

which corresponds to a negative binomial distribution with parameters r and $r / (a + r)$.

2. Let $m(i) = a / (b + i)$, $a > 0$, $b > 0$. In this case, we obtain the marginal distribution

$$f(x;\, r,\, a,\, b) = \mathcal{C}\binom{r + x - 1}{x}\frac{(a / r)_x (b + 1)_x}{(b)_x (b + a / r + 1)_x}, \; x = 0, 1, \ldots,$$

where $\mathcal{C}$, the normalization constant, is given by

$$\mathcal{C} = \frac{b(a + r(b + 1))}{b(a + r(b + 1))\, {}_1F_1(r,\, b + a / r + 1,\, a / r) + ar\, {}_1F_1(1 + r,\, 2 + b + a / r,\, a / r)}.$$

Observe that, the case $m(i) = a > 0$ corresponds with the fact that the Bayes premium, $P_B = z_n k / n + (1 - z_n) P_C$ is either equal to $k / n = \overline{X}$ (being $z_n \to 1$) or equal to P_C (being $z_n \to 0$).

Obviously, properties of these new distributions could be deeply studied.

5. Conclusions

The aim of this paper has been to illustrate some basic notions about identifiability related with credibility theory. A basic question for credibility formulas is to determine the unique relationship between these expressions and the prior distributions. Usually, it is connected with one to one correspondence between the structure function and the likelihood function, if the credibility expression is provided.

Identification of the main credibility expressions in the literature have been determined and marginal distributions, when non-credibility formulae appear, have also been given in an explicit form.

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